

Matroidal Structure of Generalized Rough Sets based on Symmetric and Transitive Relation

Bin Yang and William Zhu *

Lab of Granular Computing,
Zhangzhou Normal University, Zhangzhou 363000, China

Abstract. Rough sets are efficient for data pre-process in data mining. Lower and upper approximations are two core concepts of rough sets. This paper studies generalized rough sets based on symmetric and transitive relation from the operator-oriented view by matroidal approaches. We firstly constructs a matroidal structure of generalized rough sets based on symmetric and transitive relations, and provides an approach to study the matroid induced by a symmetric and transitive relation. Secondly, this paper establishes a close relationship between matroids and generalized rough sets. Approximation quality and roughness of generalized rough sets can be computed by the circuit of matroid theory. At last, a symmetric and transitive relation can be constructed by a matroid.

Keywords: generalized rough sets; matroid; circuit; union of matroids; upper and lower approximation

1 Introduction

Rough set theory was originally proposed by Pawlak [7,8] as a tool for dealing with the vagueness and granularity in information systems. This theory can approximately characterize an arbitrary subset of a universe by using two definable subsets called lower and upper approximation operators [1]. Now, with the fast development of rough sets in recent years, it has already been applied in fields such as knowledge discovery [2], machine learning, decision analysis, process control, pattern recognition and many other areas.

The core concepts of generalized rough sets are lower and upper approximations based on relations on a universe. There are mainly two approaches for development of the rough set theory, the constructive and axiomatic approaches. In constructive approaches, lower and upper approximation operators are constructed from the primitive notions, such as binary relations on a universe, partitions of a universe, neighborhood systems; while the axiomatic approach, which is appreciate for studying the structures of rough set algebras, takes the lower and upper approximation operators as primitive notions. By taking advantage of these two approaches the rough set theory has been combined with other mathematical theories such as fuzzy sets, algebraic theory, topology and matroids. Specifically, the establishment of matroidal structures of rough set

* Corresponding author. E-mail: williamfengzhu@gmail.com (William Zhu)

may be much helpful for some problems of rough sets such as attribute reduction and axiomatic in rough sets.

The concept of matroid was originally introduced by Whitney [3] in 1935 as a generalization of graph theory and linear algebra. The concepts and results of matroids were widely used in other fields such as algorithm design, combinatorial optimization and integer programming. Especially, the matroids provide well-established platforms for greedy algorithms. Since matroids appear in different mathematical branches, we can give explanations of matroidal structures in different mathematical backgrounds, then matroidal approaches plays a crucial role in other mathematical branches such as graph theory, linear algebra and rough sets[5].

The remainder of this paper is organized as follows: In Section 2, some basic concepts and properties related to binary relations, generalized rough sets and matroids are introduced. In Section 3, a matroidal structure of generalized rough sets based on symmetric and transitive relations is constructed. Moreover, we also explore the properties of the matroid induced by a symmetric and transitive relation. In Section 4, the lower and upper approximations of generalized rough sets based on symmetric and transitive relations are described by circuits of matroids and an approach to generate a symmetric and transitive relation by a matroid is provided. Finally, Section 5 concludes this paper.

2 Background

In this section, we review some fundamental definitions and results of generalized rough sets and matroids.

2.1 Relations on a set

In this subsection, we present some definitions and properties of binary relations used in this paper. For detailed descriptions and proofs of them, please refer to [9].

Definition 1. (Binary relation) Let U be a set, $U \times U$ the product set of U and U . Any subset R of $U \times U$ is called a binary relation on U . For any $(x, y) \in U \times U$, if $(x, y) \in R$, then we say x has relation with y , and denote this relationship as xRy .

For any $x \in U$, we call the set $\{y \in U \mid xRy\}$ the successor neighborhood of x in R and denote it as $r_R(x)$.

Throughout this paper, a binary relation is simply called a relation and it is defined on a finite and nonempty set. The relation and its properties play important roles in studying generalized rough sets.

Definition 2. (Symmetric relations) Let R be a relation on U . If for any $x, y \in U$, $y \in r_R(x) \Rightarrow x \in r_R(y)$, then we say R is symmetric.

Definition 3. (Transitive relations) Let R be a relation on U . If for any $x, y, z \in U$, $y \in r_R(x)$ and $z \in r_R(y) \Rightarrow z \in r_R(x)$, then we say R is transitive.

2.2 Generalized rough sets

In this subsection, we present some definitions and results of generalized rough sets used in this paper. In this paper, we mainly study the generalized rough sets based on symmetric and transitive relations. For detailed descriptions about generalized rough sets, please refer to [13,15,16,17,18].

Definition 4. (Approximation space) Let R be a relation on a universe U . We call the ordered pair (U, R) an approximation space.

Definition 5. (Generalized rough set) Let R be a relation on a universe U . A pair of approximation operators $\underline{R}, \overline{R}: 2^U \rightarrow 2^U$, are defined as follows: for all $X \in 2^U$,

$$\underline{R}(X) = \{x \in U \mid r_R(x) \subseteq X\},$$

$$\overline{R}(X) = \{x \in U \mid r_R(x) \cap X \neq \emptyset\}.$$

They are called the lower approximation operator and the upper approximation operator, respectively. For all $X \in 2^U$, if $\underline{R}(X) \neq \overline{R}(X)$, then X is called a R -generalized rough set. Otherwise, X is called a R -precise set.

Theorem 1. Let R be a relation on universe U . For all $X \in 2^U$,

- (1) R is symmetric $\Leftrightarrow X \subseteq \underline{R}(\overline{R}(X)) \Leftrightarrow \overline{R}(\underline{R}(X)) \subseteq X$.
- (2) R is transitive $\Leftrightarrow \underline{R}(X) \subseteq \underline{R}(\underline{R}(X)) \Leftrightarrow \overline{R}(\overline{R}(X)) \subseteq \overline{R}(X)$.

In fact, Theorem 1 reveals the relationships between the properties of relation and approximation operators. In other words, whether a relation is symmetric(transitive) or not, we could give the answer through the approximation operators.

Definition 6. (Approximation quality and roughness [7]) Let R be a relation on universe U . For all $X \in 2^U$, the approximation quality $\alpha_R(X)$ and roughness $\rho_R(X)$ of X can be defined as follows:

$$\alpha_R(X) = \frac{|\underline{R}(X)|}{|\overline{R}(X)|},$$

$$\rho_R(X) = 1 - \alpha_R(X).$$

2.3 Matroids

Matroid theory was established as a generalization, or a connection, of graph theory and linear algebra. This theory was used to study abstract relations on a subset, and it uses both of these areas of mathematics for its motivation, its basic examples, and its notation. With the rapid development in recent years, the matroid theory has always become an effective mathematic tool to study other mathematic branches. In this subsection, we present definitions, examples and results of matroids used in this paper.

Definition 7. (Matroid [4]) A matroid is an ordered pair $M = (U, \mathcal{I})$ consisting a finite set U , and a collection \mathcal{I} of subsets of U with the following three properties:

- (I1) $\emptyset \in \mathcal{I}$;
- (I2) If $I \in \mathcal{I}$, and $I' \subseteq I$, then $I' \in \mathcal{I}$;
- (I3) If $I_1, I_2 \in \mathcal{I}$, and $|I_1| < |I_2|$, then there exists $e \in I_2 - I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$, where $|I|$ denotes the cardinality of I .

Any element of \mathcal{I} is called an independent set.

Example 1. Let $U = \{a, b, c\}$, $\mathcal{I}_1 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$ and $\mathcal{I}_2 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$. Clearly, \mathcal{I}_1 and \mathcal{I}_2 satisfy the independent set axioms of matroids. Hence (U, \mathcal{I}_1) and (U, \mathcal{I}_2) are matroids, respectively.

Definition 8. (Dependent set [4]) Let $M = (U, \mathcal{I})$ be a matroid. If $X \notin \mathcal{I}$, then we say X is a dependent set of M .

Definition 9. (Circuit [4]) Let $M = (U, \mathcal{I})$ be a matroid. A minimal dependent set in M is called a circuit of M , and we denote the family of all circuits of M by $\mathcal{C}(M)$.

Example 2. (Continued from Example 1) The family of dependent sets of (U, \mathcal{I}_1) is $\{\{a, b\}, \{a, b, c\}\}$ and the family of dependent sets of (U, \mathcal{I}_2) is $\{\{a, c\}, \{a, b, c\}\}$. Hence, the family of circuits of (U, \mathcal{I}_1) and (U, \mathcal{I}_2) are $\mathcal{C}_1 = \{\{a, b\}\}$ and $\mathcal{C}_2 = \{\{a, c\}\}$, respectively.

Theorem 2. (Circuit axiom [4]) Let $M = (U, \mathcal{I})$ be a matroid and $\mathcal{C} = \mathcal{C}(M)$, then \mathcal{C} satisfies the following three properties:

(C1) $\emptyset \notin \mathcal{C}$;

(C2) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$;

(C3) If $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$ and $e \in C_1 \cap C_2$, then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - \{e\}$.

Theorem 3. [4] Let U be a nonempty and finite set and \mathcal{C} a family of subsets of U . If \mathcal{C} satisfies (C1), (C2) and (C3) of Theorem 2, then there exists $M = (U, \mathcal{I})$ such that $\mathcal{C} = \mathcal{C}(M)$.

According to the above theorem, we see a matroid can be determined by its circuits.

Definition 10. (Normal matroid [4]) Let $M = (U, \mathcal{I})$ be a matroid. If $\bigcup \mathcal{I} = U$, then we call M is a normal matroid.

Example 3. (Continued from Example 1) It is easy to prove the following results:

$$\begin{aligned}\bigcup \mathcal{I}_1 &= \{a, b, c\} = U; \\ \bigcup \mathcal{I}_2 &= \{a, b, c\} = U.\end{aligned}$$

Hence, the matroids (U, \mathcal{I}_1) and (U, \mathcal{I}_2) are two normal matroids.

Definition 11. (Union of matroids [4]) Let $M_1 = (U, \mathcal{I}_1)$ and $M_2 = (U, \mathcal{I}_2)$ be two matroids. Then we define the union of M_1 and M_2 as follows:

$$M_1 + M_2 = (U, \mathcal{I}_1 + \mathcal{I}_2), \text{ where } \mathcal{I}_1 + \mathcal{I}_2 = \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}.$$

Theorem 4. [4] If M_1 and M_2 are two matroids on U , then $M_1 + M_2$ is a matroid.

Example 4. (Continued from Example 1) According to Example 1 and Definition 11, it is easy to compute

$$\mathcal{I}_1 + \mathcal{I}_2 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Clearly, $\mathcal{I}_1 + \mathcal{I}_2$ satisfies the axiom of matroid, i.e., $(U, \mathcal{I}_1 + \mathcal{I}_2)$ is a matroid.

3 Structure of generalized rough sets

In this section, we establish a matroidal structure for generalized rough sets based on a symmetric and transitive relation, and study some properties of the matroidal structure. In [6,10,11,12] some approaches to generate matroidal structures of Pawlak's rough sets and covering-based rough sets were proposed, respectively. Now, we propose an approach to generate a matroidal structure of generalized rough sets based on symmetric and transitive relations.

Definition 12. Let R be a symmetric and transitive relation on U . We can define $\mathcal{C}(R)$ as follows: for all $x, y \in U$,

$$\mathcal{C}(R) = \{\{x, y\} \mid x \neq y, y \in r_R(x)\}.$$

Proposition 1. If R is a symmetric and transitive relation on U , then $\mathcal{C}(R)$ satisfies the circuit axioms of matroids.

Proof. We need to prove $\mathcal{C}(R)$ satisfies (C1), (C2) and (C3) of Theorem 2. According to Definition 12, for all $C \in \mathcal{C}(R)$, $|C| = 2$. Thus $\emptyset \notin \mathcal{C}(R)$ and (C1) holds. For all $C_1, C_2 \in \mathcal{C}(R)$, $|C_1| = |C_2| = 2$. If $C_1 \subseteq C_2$, then $C_1 = C_2$ and (C2) holds. Let $C_1, C_2 \in \mathcal{C}(R)$, $C_1 \neq C_2$ and $e \in C_1 \cap C_2$. Without losing generality, let $C_1 = \{x_1, e\}$ and $C_2 = \{x_2, e\}$, where $x_1 \neq x_2$. According to Definition 12, $e \in r_R(x_1)$ and $e \in r_R(x_2)$, since R is symmetric and transitive, then $x_1 \in r_R(x_2)$. Suppose $C_3 = \{x_1, x_2\}$, then $C_3 \in \mathcal{C}(R)$ and $C_3 \subseteq (C_1 \cup C_2) - \{e\}$. Thus $\mathcal{C}(R)$ satisfies (C3). To sum up, $\mathcal{C}(R)$ satisfies the circuit axioms of matroids.

Suppose there exists a matroid M such that $\mathcal{C}(M) = \mathcal{C}(R)$, then we say $M(R) = (U, \mathcal{I}(R))$ is the induced matroid by R .

In fact, according to Theorem 3, Definition 12 and Proposition 1, we find that a symmetric and transitive relation on a universe determines a matroid. In other words, a matroidal structure of generalized rough sets based on symmetric and transitive relations can be constructed.

Example 5. Let $U = \{a, b, c\}$ and $R = \{(a, a), (a, b), (b, a), (b, b)\}$ a symmetric and transitive relation on U . According to Proposition 1, thus there exists a matroid $M(R) = (U, \mathcal{I}(R))$ induced by R , where

$$\begin{aligned} \mathcal{C}(R) &= \{\{a, b\}\}, \\ \mathcal{I}(R) &= \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}. \end{aligned}$$

Proposition 2. If R be a symmetric and transitive relation on U and $M(R) = (U, \mathcal{I}(R))$ the matroid induced by R , then $M(R)$ is a normal matroid.

Proof. According to Proposition 1, for all $C \in \mathcal{C}(R)$, then $|C| = 2$. In other words, for all x , then $\{x\} \in \mathcal{I}(R)$. Thus $\bigcup \mathcal{I}(R) = U$, i.e., $M(R)$ is a normal matroid.

Example 6. (Continued from Example 5) The matroid induced by R is $M(R) = (U, \mathcal{I}(R))$, where $\mathcal{I}(R) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$. Since $\bigcup \mathcal{I}(R) = \{a, b, c\} = U$, thus $M(R)$ is a normal matroid.

Proposition 3. Let R be a symmetric and transitive relation on U and $M(R) = (U, \mathcal{I}(R))$ the matroid induced by R . If $C_1, C_2 \in \mathcal{C}(R)$, $e_1 \in C_1 - C_2$, $e_2 \in C_2 - C_1$ and $C_1 \cap C_2 \neq \emptyset$, then there exists $C_3 \in \mathcal{C}(R)$ such that $e_1, e_2 \in C_3 \subseteq C_1 \cup C_2$.

Proof. Let $C_1, C_2 \in \mathcal{C}(R)$. Without losing generality, let $C_1 = \{x, e_1\}$ and $C_2 = \{x, e_2\}$, where $e_1 \notin C_2$, $e_2 \notin C_1$ and $x \neq e_1 \neq e_2$. Since R is a symmetric and transitive relation, thus $e_2 \in r_R(e_1)$. Suppose $C_3 = \{e_1, e_2\}$, then $C_3 \in \mathcal{C}(R)$ and $e_1, e_2 \in C_3 \subseteq C_1 \cup C_2$.

Proposition 4. Let R_1 and R_2 be two symmetric and transitive relations on U . Let $M(R_1)$, $M(R_2)$ and $M(R_1 \cap R_2)$ be the matroids induced by R_1 , R_2 and $R_1 \cap R_2$, respectively. Then $\mathcal{I}(R_1) \subseteq \mathcal{I}(R_1 \cap R_2)$ and $\mathcal{I}(R_2) \subseteq \mathcal{I}(R_1 \cap R_2)$.

Proof. Since $R_1 \cap R_2 \subseteq R_1$ and $R_1 \cap R_2 \subseteq R_2$, according to Definition 12, then $\mathcal{C}(R_1 \cap R_2) \subseteq \mathcal{C}(R_1)$ and $\mathcal{C}(R_1 \cap R_2) \subseteq \mathcal{C}(R_2)$. According to Definition 8 and Definition 9, $\mathcal{C}(R_1) \subseteq 2^U - \mathcal{I}(R_1)$, $\mathcal{C}(R_2) \subseteq 2^U - \mathcal{I}(R_2)$ and $\mathcal{C}(R_1 \cap R_2) \subseteq 2^U - \mathcal{I}(R_1 \cap R_2)$, therefore $\mathcal{I}(R_1) \subseteq \mathcal{I}(R_1 \cap R_2)$ and $\mathcal{I}(R_2) \subseteq \mathcal{I}(R_1 \cap R_2)$.

Example 7. Let $U = \{a, b, c, d\}$ be a universe.

Let $R_1 = \{(a, a), (a, b), (b, a), (b, b), (a, c), (c, a), (b, c), (c, b), (c, c)\}$ and $R_2 = \{(a, a), (a, b), (b, a), (b, b), (a, d), (d, a), (b, d), (d, b), (d, d)\}$ be two symmetric and transitive relations on U , respectively.

Then $R_1 \cap R_2 = \{(a, a), (a, b), (b, a), (b, b)\}$ is a symmetric and transitive relation. $M(R_1) = (U, \mathcal{I}(R_1))$, $M(R_2) = (U, \mathcal{I}(R_2))$ and $M(R_1 \cap R_2) = (U, \mathcal{I}(R_1 \cap R_2))$ are induced by R_1 , R_2 and $R_1 \cap R_2$, respectively. Then

$\mathcal{I}(R_1) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}\};$

$\mathcal{I}(R_2) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{b, c\}, \{c, d\}\};$

$\mathcal{I}(R_1 \cap R_2) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}, \{a, c, d\}\}.$

Hence, $\mathcal{I}(R_1) \subseteq \mathcal{I}(R_1 \cap R_2)$ and $\mathcal{I}(R_2) \subseteq \mathcal{I}(R_1 \cap R_2)$.

In fact, the intersection of symmetric and transitive relations is also a symmetric and transitive relation. Hence, a matroid can be generated by the intersection of symmetric and transitive relations.

Example 8. Let $U = \{a, b, c\}$ and $R_1 = \{(a, a), (a, b), (b, a), (b, b)\}$, $R_2 = \{(a, a), (a, c), (c, a), (c, c)\}$ be two symmetric and transitive relations on U . Then $M(R_1) = (U, \mathcal{I}(R_1))$ and $M(R_2) = (U, \mathcal{I}(R_2))$ are the two matroids induced by R_1 and R_2 , respectively. Where $\mathcal{I}(R_1) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$, $\mathcal{I}(R_2) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$. Hence, $M(R_1) + M(R_2) = (U, \mathcal{I}(R_1) + \mathcal{I}(R_2))$ is a matroid and $\mathcal{I}(R_1) + \mathcal{I}(R_2) = 2^U$.

Example 9. (Continued from Example 8) Let $M(R_1) \times M(R_2) = (U, \mathcal{I}(R_1) \cap \mathcal{I}(R_2))$. Then $\mathcal{I}(R_1) \cap \mathcal{I}(R_2) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}\}$. According to Definition 7, it does not satisfy (I3), thus ordered pair $(U, \mathcal{I}(R_1) \cap \mathcal{I}(R_2))$ is not a matroid.

The above example indicates that the intersection of independent set families of two matroids may be not an independent set family of a matroid.

4 Generalized rough sets based on matroids

The lower and upper approximation operators are the core concepts of rough sets. In [14] and [19], the authors proved the existence and uniqueness of a certain binary relation for an algebraic operator with special properties. In this section, for a symmetric and transitive relation on a universe, we use the circuits of the matroid induced by the relation to represent the lower and upper approximations of the generalized rough sets.

Proposition 5. *Let R be a symmetric and transitive relation on U and $M(R) = (U, \mathcal{I}(R))$ the matroid induced by R . For all $X \in 2^U$,*

$$\begin{aligned}\overline{R}(X) &= (\bigcup\{C \in \mathcal{C}(R) \mid C \cap X \neq \emptyset\}) \cup \{x \in X \mid x \in r_R(x)\}, \\ \underline{R}(X) &= (\bigcup\{C \in \mathcal{C}(R) \mid \forall x \forall y (x \in C \wedge \{x, y\} \in \mathcal{C}(R) \rightarrow \{x, y\} \subseteq X)\}) \cup \{x \in X \mid x \in r_R(x)\} \cup \{x \in X \mid r_R(x) = \emptyset\}.\end{aligned}$$

Proof. According to Definition 1, we need to prove (1) $\{x \in U \mid r_R(x) \cap X \neq \emptyset\} = (\bigcup\{C \in \mathcal{C}(R) \mid C \cap X \neq \emptyset\}) \cup \{x \in X \mid x \in r_R(x)\}$ and (2) $\{x \in U \mid r_R(x) \subseteq X\} = (\bigcup\{C \in \mathcal{C}(R) \mid \forall x \forall y (x \in C \wedge \{x, y\} \in \mathcal{C}(R) \rightarrow \{x, y\} \subseteq X)\}) \cup \{x \in X \mid x \in r_R(x)\} \cup \{x \in X \mid r_R(x) = \emptyset\}$ hold.

(1) On one hand, if for any $x_1 \in (\bigcup\{C \in \mathcal{C}(R) \mid C \cap X \neq \emptyset\}) \cup \{x \in X \mid x \in r_R(x)\}$, then $x_1 \in \{x \in X \mid x \in r_R(x)\}$ or $x_1 \in \bigcup\{C \in \mathcal{C}(R) \mid C \cap X \neq \emptyset\}$. If $x_1 \in \{x \in X \mid x \in r_R(x)\}$, then $x_1 \in X$ and $x_1 \in r_R(x_1)$, i.e., $x_1 \in \{x \in U \mid r_R(x) \cap X \neq \emptyset\}$; if $x_1 \in \bigcup\{C \in \mathcal{C}(R) \mid C \cap X \neq \emptyset\}$, then there exists $x_2 \neq x_1$ such that $\{x_1, x_2\} \in \mathcal{C}(R)$ and $\{x_1, x_2\} \cap X \neq \emptyset$. Hence, $r_R(x_1) \cap X \neq \emptyset$. Thus $(\bigcup\{C \in \mathcal{C}(R) \mid C \cap X \neq \emptyset\}) \cup \{x \in X \mid x \in r_R(x)\} \subseteq \{x \in U \mid r_R(x) \cap X \neq \emptyset\}$. On the other hand, for any $x_1 \in \{x \in U \mid r_R(x) \cap X \neq \emptyset\}$, there exists $x_2 \in X$ such that $x_2 \in r_R(x_1)$. If $x_1 \neq x_2$, then $\{x_1, x_2\} \in \mathcal{C}(R)$ and $x_1 \in \bigcup\{C \in \mathcal{C}(R) \mid C \cap X \neq \emptyset\}$; if $x_1 = x_2$, then $x_1 \in \{x \in X \mid x \in r_R(x)\}$. Therefore, $\{x \in U \mid r_R(x) \cap X \neq \emptyset\} = (\bigcup\{C \in \mathcal{C}(R) \mid C \cap X \neq \emptyset\}) \cup \{x \in X \mid x \in r_R(x)\}$ holds.

(2) On one hand, for any $x_1 \in (\bigcup\{C \in \mathcal{C}(R) \mid \forall x \forall y (x \in C \wedge \{x, y\} \in \mathcal{C}(R) \rightarrow \{x, y\} \subseteq X)\}) \cup \{x \in X \mid x \in r_R(x)\} \cup \{x \in X \mid r_R(x) = \emptyset\}$, $x_1 \in \bigcup\{C \in \mathcal{C}(R) \mid \forall x \forall y (x \in C \wedge \{x, y\} \in \mathcal{C}(R) \rightarrow \{x, y\} \subseteq X)\}$ or $x_1 \in \{x \in X \mid x \in r_R(x)\}$ or $x_1 \in \{x \in X \mid r_R(x) = \emptyset\}$. If $x_1 \in \bigcup\{C \in \mathcal{C}(R) \mid \forall x \forall y (x \in C \wedge \{x, y\} \in \mathcal{C}(R) \rightarrow \{x, y\} \subseteq X)\}$ and for all $\{x_1, x_2\} \in \mathcal{C}(R)$, then $\{x_1, x_2\} \subseteq X$; if $x_1 \in \{x \in X \mid x \in r_R(x)\}$, then $x_1 \in r_R(x_1)$; if $x_1 \in \{x \in X \mid r_R(x) = \emptyset\}$, then $r_R(x_1) \subseteq X$ and $x_1 \in \{x \in U \mid r_R(x) \subseteq X\}$. Thus $r_R(x_1) \subseteq X$, i.e., $x_1 \in \{x \in U \mid r_R(x) \subseteq X\}$. Therefore, $(\bigcup\{C \in \mathcal{C}(R) \mid \forall x \forall y (x \in C \wedge \{x, y\} \in \mathcal{C}(R) \rightarrow \{x, y\} \subseteq X)\}) \cup \{x \in X \mid x \in r_R(x)\} \subseteq \{x \in U \mid r_R(x) \subseteq X\}$ holds. On the other hand, for any $x_1 \in \{x \in U \mid r_R(x) \subseteq X\}$, $r_R(x_1) \subseteq X$. For all $x_2 \in r_R(x_1)$, if $x_1 = x_2$, then $x_1 \in \{x \in X \mid x \in r_R(x)\}$; if $x_1 \neq x_2$, then $\{x_1, x_2\} \in \mathcal{C}(R)$ and $\{x_1, x_2\} \subseteq X$, i.e., $x_1 \in \bigcup\{C \in \mathcal{C}(R) \mid \forall x \forall y (x \in C \wedge \{x, y\} \in \mathcal{C}(R) \rightarrow \{x, y\} \subseteq X)\}$. Therefore, $\{x \in U \mid r_R(x) \subseteq X\} = (\bigcup\{C \in \mathcal{C}(R) \mid \forall x \forall y (x \in C \wedge \{x, y\} \in \mathcal{C}(R) \rightarrow \{x, y\} \subseteq X)\}) \cup \{x \in X \mid x \in r_R(x)\}$ holds.

To sum up, we have already finished the proof of this proposition.

The above proposition presents that the lower and upper approximation operators of generalized rough sets based on symmetric and transitive relations can be described

by the circuits of matroids. Therefore, we can compute the approximation quality and roughness of generalized rough sets by the circuits of matroid.

Example 10. Let $U = \{a, b, c, d, e, f\}$ be a universe and $R = \{(a, a), (a, b), (b, a), (b, b), (a, d), (d, a), (b, d), (d, b), (d, d), (c, c), (e, e)\}$ a symmetric and transitive relation on U . Suppose $X_1 = \{a, b, c, e, f\}$ and $X_2 = \{a, c, d\}$, then the lower and upper approximations, approximation quality and roughness of X_1 and X_2 could be computed as follows, respectively.

(1) According to Definition 5 and Definition 6, since $r_R(a) = \{a, b, d\}$, $r_R(b) = \{a, b, d\}$, $r_R(c) = \{c\}$, $r_R(d) = \{a, b, d\}$, $r_R(e) = \{e\}$, $r_R(f) = \emptyset$, then $\underline{R}(X_1) = \{c, e, f\}$, $\overline{R}(X_1) = \{a, b, c, d, e\}$, $\alpha_R(X_1) = \frac{|\underline{R}(X_1)|}{|\overline{R}(X_1)|} = 0.6$, $\rho_R(X_1) = 1 - \alpha_R(X_1) = 0.4$; $\underline{R}(X_2) = \{c, f\}$, $\overline{R}(X_2) = \{a, b, c, d\}$, $\alpha_R(X_2) = \frac{|\underline{R}(X_2)|}{|\overline{R}(X_2)|} = 0.5$, $\rho_R(X_2) = 1 - \alpha_R(X_2) = 0.5$.

(2) According to Proposition 5 and Definition 6, since $\mathcal{C}(R) = \{\{a, b\}, \{a, d\}, \{b, d\}\}$, then $\underline{R}(X_1) = \{c, e, f\}$, $\overline{R}(X_1) = \{a, b, c, d, e\}$, $\alpha_R(X_1) = \frac{|\underline{R}(X_1)|}{|\overline{R}(X_1)|} = 0.6$, $\rho_R(X_1) = 1 - \alpha_R(X_1) = 0.4$; $\underline{R}(X_2) = \{c, f\}$, $\overline{R}(X_2) = \{a, b, c, d\}$, $\alpha_R(X_2) = \frac{|\underline{R}(X_2)|}{|\overline{R}(X_2)|} = 0.5$, $\rho_R(X_2) = 1 - \alpha_R(X_2) = 0.5$.

In fact, according to Proposition 5, we know the lower and upper approximation operators of generalized rough sets based on symmetric and transitive relations can be described by the circuits of matroids induced by symmetric and transitive relations. Thus according to Theorem 1, we can describe some properties of a symmetric and transitive relation by the circuits of the matroid induced by this relation.

Definition 13. Let $M = (U, \mathcal{I})$ be a matroid. We can define a relation $R(M)$ as follows:

$$R(M) = \{(x, y) \subseteq U \times U \mid \{x, y\} \in \mathcal{C}(M)\} \cup \{(x, x), (y, y)\}.$$

Proposition 6. If $M = (U, \mathcal{I})$ is a matroid, then the relation $R(M)$ is symmetric and transitive.

Proof. According to Definition 13, for all $C = \{x_1, y_1\} \in \mathcal{C}(M)$, thus $(x_1, y_1) \in R$. Since $\{x_1, y_1\} = \{y_1, x_1\}$, then $(y_1, x_1) \in R$ and symmetry of R holds. For all $\{x_1, y_1\} \in \mathcal{C}(M)$ and $\{y_1, z_1\} \in \mathcal{C}(M)$, according to the (C3) of Theorem 2, thus $(x_1, z_1) \in R$. Therefore, the transitivity of R satisfies.

In fact, the above proposition shows how to construct a symmetric and transitive relation by a matroid.

Example 11. Let $U = \{a, b, c, d, e, f\}$ be a universe and $M = (U, \mathcal{I})$ a matroid on U . If $\mathcal{C}(M) = \{\{a, b\}, \{a, c\}, \{a, e\}, \{b, c\}, \{c, f\}, \{e, f\}\}$, according to Proposition 7, then the symmetric and transitive relation $R(M)$ induced by M is as follows:

$$R(M) = \{(a, a), (b, b), (c, c), (e, e), (f, f), (a, b), (b, a), (a, c), (c, a), (a, e), (e, a), (b, c), (c, b), (c, f), (f, c), (e, f), (f, e)\}.$$

In Section 3, we have already proved the union of matroids is a matroid. Now, in following proposition we will explore the relationships between the symmetric and transitive relation generated by the union of two matroids and the two relations generated by these two matroids, respectively.

Proposition 7. *Let R_1 and R_2 be two symmetric and transitive relations on U . If the matroids $M(R_1)$ and $M(R_2)$ were induced by R_1 and R_2 , respectively, then $R(M(R_1) + M(R_2))$ is an empty relation.*

Proof. According to Definition 11 and Definition 13, it is easy to prove $\mathcal{C}(M(R_1) + M(R_2)) = \emptyset$. Hence, $R(M(R_1) + M(R_2))$ is an empty relation.

Example 12. (Continued from Example 8) According to Example 8, we have already known $\mathcal{I}(M(R_1) + M(R_2)) = 2^U$. Therefore, $\mathcal{C}(M(R_1) + M(R_2)) = \emptyset$ and $R(M(R_1) + M(R_2))$ is a empty relation.

5 Conclusions

In this paper we construct the matroidal structure of a symmetric and transitive relation on a nonempty and finite set. Firstly, we use properties and results of the generalized rough sets to study the properties of the matroid induced by a symmetric and transitive relation. Secondly, we represent the lower approximation operator and upper approximation operator by the circuits of the matroid induced by a symmetric and transitive relation. Finally, a symmetric and transitive relation can be generated by a matroid.

6 Acknowledgments

This work is supported in part by the National Natural Science Foundation of China under Grant No. 61170128, the Natural Science Foundation of Fujian Province, China, under Grant Nos. 2011J01374 and 2012J01294, and the Science and Technology Key Project of Fujian Province, China, under Grant No. 2012H0043.

References

1. Chen, D., Zhang, W., Yeung, D., Tsang, E.: Rough approximations on a complete completely distributive lattice with applications to generalized rough sets. *Information Sciences* **176** (2006) 1829–1848
2. Gálvez, J.F., Diaz, F., Carrión, P., Garcia, A.: An application for knowledge discovery based on a revision of vprs model. In: *Rough Sets and Current Trends in Computing*. Volume 2005 of LNCS. (2000) 296–303
3. H. Whitney: On the abstract properties of linear dependence. *American Journal of Mathematics* **57** (1935) 509–533
4. Lai, H.: *Matroid theory*. Higher Education Press, Beijing (2001)
5. Li, X., Liu, S.: Matroidal approaches to rough set theory via closure operators. *International Journal of Approximate Reasoning* **53** (2012) 513–527

6. Liu, Y., Zhu, W., Zhang, Y.: Relationship between partition matroid and rough set through k-rank matroid. *Journal of Information and Computational Science* **8** (2012) 2151–2163
7. Pawlak, Z.: *Rough sets: theoretical aspects of reasoning about data*. Kluwer Academic Publishers, Boston (1991)
8. Pawlak, Z.: Rough sets. *International Journal of Computer and Information Sciences* **11** (1982) 341–356
9. Rajagopal, P., Masone, J.: *Discrete mathematics for computer science*. Saunders College, Canada (1992)
10. Wang, S., Zhu, Q., Zhu, W., Min, F.: Matroidal structure of rough sets and its characterization to attribute reduction. to appear in *Knowledge-Based Systems* (2012)
11. Wang, S., Zhu, Q., Zhu, W., Min, F.: Quantitative analysis for covering-based rough sets through the upper approximation number. to appear in *Information Sciences* (2012)
12. Wang, S., Zhu, W.: Matroidal structure of covering-based rough sets through the upper approximation number. *International Journal of Granular Computing, Rough Sets and Intelligent Systems* **2** (2011) 141–148
13. Yang, X., Li, T.: The minimization of axiom sets characterizing generalized approximation operators. *Information Sciences* **176** (2006) 887–899
14. Yao, Y.: Constructive and algebraic methods of theory of rough sets. *Information Sciences* **109** (1998) 21–47
15. Zhu, W., Wang, F.: Reduction and axiomization of covering generalized rough sets. *Information Sciences* **152** (2003) 217–230
16. Zhu, W., Wang, F.: Axiomatic systems of generalized rough sets. In: *Rough Set and Knowledge Technology*. Volume 4062 of LNAI. (2006) 216–221
17. Zhu, W.: Generalized rough sets based on relations. *Information Sciences* **177** (2007) 4997–5011
18. Zhu, W.: Relationship between generalized rough sets based on binary relation and covering. *Information Sciences* **179** (2009) 210–225
19. Zhu, W., Wang, F.: Binary relation based rough set. In: *Fuzzy Systems and Knowledge Discovery*. Volume 4223 of LNAI. (2006) 276–285